Electrodynamics in Uniformly Rotating Frames as Viewed from an Inertial Frame

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Abstract- In the current paper we present a generalization of the transforms of the electromagnetic field from an inertial frame of reference into the frame co-moving with a uniformly rotating particle. The solution is of great interest for real time applications, because earth-bound laboratories are inertial only in approximation. We conclude by deriving the general form of the relativistic Doppler effect and of the relativistic aberration formulas for the case of uniformly rotating frames.

Key Words: Accelerated motion, General coordinate transformations, Accelerated particles, Planar electromagnetic waves, Relativistic Doppler effect, Relativistic aberration.

1. INTRODUCTION

Real life applications include accelerating and rotating frames more often than the idealized case of inertial frames. Our daily experiments happen in the laboratories attached to the rotating, continuously accelerating Earth. Many books and papers have been dedicated to transformations between particular cases of rectilinear acceleration and/or rotation [1] and to the applications of such formulas [2-11]. The main idea of this paper is to generate a standard blueprint for a general solution that gives equivalent of the Lorentz transforms for the case of the transforms between an inertial frame and a rotating frame.

2. UNIFORMLY ROTATING MOTION–THE TRANSFORMS OF THE ELECTROMAGNETIC FIELD

In this section we discuss the case of the particle moving in an arbitrary plane, with the normal given by the constant angular velocity *ω(a, b, c)* (Fig. 2). According to Moller [1], the simpler case when *ω* is aligned with the z-axis produces the transformation between the rotating frame S'(τ) attached to the particle and an inertial, non-rotating frame S attached to the center of rotation:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = Phy\_rotation \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{1}$$

where

$$Phy\_rotation =$$

$$\begin{bmatrix} \cos\alpha\cos\beta + \gamma\sin\alpha\sin\beta & \mathrm{sim}\,\alpha\cos\beta - \gamma\cos\alpha\sin\beta & 0 & u\gamma\sin\beta \\ \cos\alpha\sin\beta - \gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta & 0 & -u\gamma\cos\beta \\ 0 & 0 & 1 & 0 \\ \frac{\gamma\sin\alpha\sin\beta}{c^2} & -\frac{\gamma\sin\alpha\sin\beta}{c^2} & 0 & \gamma \end{bmatrix} \tag{2}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}, \quad u = r\omega, \quad \alpha = \omega / \tau, \quad \beta = \omega\gamma^2\tau$$

In previous papers, we have derived the transformations from the rotating frame S' into the inertial frame S:

$$\begin{aligned} E_x &= (\gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta)E'_x \\ &+ (\gamma\sin\alpha\cos\beta - \cos\alpha\sin\beta)E'_y - u\gamma\cos\beta B'_z \\ E_y &= (\gamma\cos\alpha\sin\beta - \sin\alpha\cos\beta)E'_x \\ &+ (\gamma\sin\alpha\sin\beta + \cos\alpha\cos\beta)E'_y - u\gamma\sin\beta B'_z \\ B_z &= -\frac{u\gamma\cos\alpha}{c^2}E'_x - \frac{u\gamma\sin\alpha}{c^2} + \gamma B'_z \\ B_x &= (\gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta)B'_x \\ &+ (\gamma\sin\alpha\cos\beta - \cos\alpha\sin\beta)B'_y - \frac{u\gamma\cos\beta}{c^2}E'_z \\ B_y &= (\gamma\cos\alpha\sin\beta - \sin\alpha\cos\beta)B'_x \\ &+ (\gamma\sin\alpha\sin\beta + \cos\alpha\cos\beta)B'_y - \frac{u\gamma\sin\beta}{c^2}E'_z \\ E_z &= -u\gamma\cos\alpha B'_x - u\gamma\sin\alpha B'_y + \gamma E'_z \end{aligned} \tag{3}$$

The above can be re-cast in matrix form

$$\begin{bmatrix} E_x \\ E_y \\ B_z \end{bmatrix} = \begin{bmatrix} \gamma\cos\alpha\cos\beta+\sin\alpha\sin\beta & \gamma\sin\alpha\cos\beta-\cos\alpha\sin\beta & -u\gamma\cos\beta \\ \gamma\cos\alpha\sin\beta-\sin\alpha\cos\beta & \gamma\sin\alpha\sin\beta+\cos\alpha\cos\beta & -u\gamma\sin\beta \\ -\frac{u\gamma\cos\alpha}{c^2} & -\frac{u\gamma\sin\alpha}{c^2} & \gamma \end{bmatrix} \begin{bmatrix} E'_x \\ E'_y \\ B'_z \end{bmatrix} \tag{4}$$

$$\begin{bmatrix} B_x \\ B_y \\ E_z \end{bmatrix} = \begin{bmatrix} \gamma\cos\alpha\cos\beta+\sin\alpha\sin\beta & \gamma\sin\alpha\cos\beta-\cos\alpha\sin\beta & -\frac{u\gamma\cos\beta}{c^2} \\ \gamma\cos\alpha\sin\beta-\sin\alpha\cos\beta & \gamma\sin\alpha\sin\beta+\cos\alpha\cos\beta & -\frac{u\gamma\sin\beta}{c^2} \\ -u\gamma\cos\alpha & -u\gamma\sin\alpha & \gamma \end{bmatrix} \begin{bmatrix} B'_x \\ B'_y \\ E'_z \end{bmatrix} \tag{5}$$

This observation allows us an easy way of inverting the transforms in order to obtain the transforms from the inertial frame S into the rotating frame S':

$$\begin{bmatrix} E'_x \\ E'_y \\ B'_z \end{bmatrix} = \begin{bmatrix} \gamma\cos\alpha\cos\beta+\sin\alpha\sin\beta & \gamma\cos\alpha\sin\beta-\sin\alpha\cos\beta & u\gamma\cos\alpha \\ \gamma\sin\alpha\cos\beta-\cos\alpha\sin\beta & \gamma\sin\alpha\sin\beta+\cos\alpha\cos\beta & u\gamma\sin\alpha \\ \frac{u\gamma\cos\beta}{c^2} & \frac{u\gamma\sin\beta}{c^2} & \gamma \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ B_z \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} B'_x \\ B'_y \\ E'_z \end{bmatrix} = \begin{bmatrix} \gamma\cos\alpha\cos\beta+\sin\alpha\sin\beta & \gamma\cos\alpha\sin\beta-\sin\alpha\cos\beta & \frac{u\gamma\cos\alpha}{c^2} \\ \gamma\sin\alpha\cos\beta-\cos\alpha\sin\beta & \gamma\sin\alpha\sin\beta+\cos\alpha\cos\beta & \frac{u\gamma\sin\alpha}{c^2} \\ u\gamma\cos\beta & u\gamma\sin\beta & \gamma \end{bmatrix} \begin{bmatrix} B_x \\ B_y \\ E_z \end{bmatrix} \tag{7}$$

A very nice consequence of (6) and (7) is that:

$$E'^2 - c^2 B'^2 = E^2 - c^2 B^2 \tag{8}$$

The rotation "mixes" the components of the electromagnetic tensor in a way that is different from the cases of inertial motion or uniformly accelerated motion.

3. PLANAR WAVE TRANSFORMATION AND SPEED OF LIGHT IN A UNIFORMLY ROTATING FRAME

In this section we apply the formalism derived in the previous paragraph in order to obtain the transform of a planar wave. Assume that a planar wave is propagating along the y axis in the inertial frame S. The wave has the electric component E_x and the magnetic component B_z along the x and z axes, respectively. The components equations are (Fig. 1):

$$\begin{aligned} \mathbf{E}_x &= E_{0x}\cos(\Omega t - k_y y + \theta)\mathbf{e}_x \\ \mathbf{B}_z &= B_{0z}\cos(\Omega t - k_y y + \theta)\mathbf{e}_z \end{aligned} \tag{9}$$

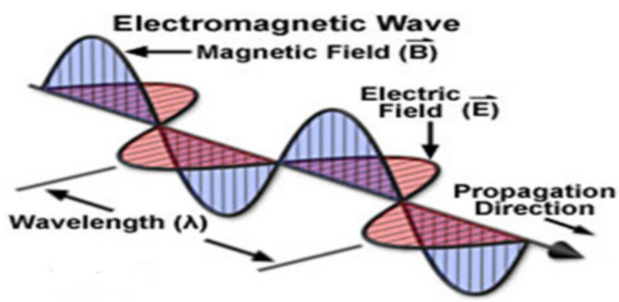

Fig. 1: The electromagnetic wave

On the other hand:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{bmatrix} b_{11} & b_{12} & 0 & b_{14} \\ b_{21} & b_{22} & 0 & b_{24} \\ 0 & 0 & 1 & 0 \\ b_{41} & b_{42} & 0 & b_{44} \end{bmatrix} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{10}$$

where:

$$\begin{bmatrix} b_{11} & b_{12} & 0 & b_{14} \\ b_{21} & b_{22} & 0 & b_{24} \\ 0 & 0 & 1 & 0 \\ b_{41} & b_{41} & 0 & b_{44} \end{bmatrix} =$$

$$\begin{bmatrix} \cos\alpha\cos\beta + \gamma\sin\alpha\sin\beta & \mathrm{sim}\alpha\cos\beta - \gamma\cos\alpha\sin\beta & 0 & u\gamma\sin\beta \\ \cos\alpha\sin\beta - \gamma\sin\alpha\cos\beta & \sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta & 0 & -u\gamma\cos\beta \\ 0 & 0 & 1 & 0 \\ \dfrac{\gamma\sin\alpha\sin\beta}{c^2} & -\dfrac{\gamma\sin\alpha\sin\beta}{c^2} & 0 & \gamma \end{bmatrix} \tag{11}$$

Substituting (10) into (9) we obtain:

$$\begin{aligned} E_x &= E_{0x}\cos[\Omega(b_{41}x' + b_{42}y' + b_{44}t') - k_y(b_{21}x' + b_{22}y' + b_{24}t') + \theta] = \\ &= E_{0x}\cos[(\Omega b_{44} - k_y b_{24})t' - (k_y b_{22} - \Omega b_{42})y' - (k_y b_{21} - \Omega b_{41})x' + \theta] \\ E_y &= E_z = 0 \\ B_z &= B_{0z}\cos[(\Omega b_{44} - k_y b_{24})t' - (k_y b_{22} - \Omega b_{42})y' - (k_y b_{21} - \Omega b_{41})x' + \theta] \\ B_x &= B_y = 0 \end{aligned} \tag{12}$$

On the other hand, in frame S', the wave equation has two alternative forms, First one:

$$\begin{aligned} E_x' &= (\gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta)E_x + (u\gamma\cos\alpha)B_z\} = \\ &= [(\gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta)E_{0x} + (u\gamma\cos\alpha)B_{0z}]* \\ &*\cos[(\Omega b_{44} - k_y b_{24})t' - (k_y b_{22} - \Omega b_{42})y' - (k_y b_{21} - \Omega b_{41})x' + \theta] \\ E_y' &= [(\gamma\sin\alpha\cos\beta - \cos\alpha\sin\beta)E_{0x} + (u\gamma\sin\alpha)B_{0z}]* \\ &*\cos[(\Omega b_{44} - k_y b_{24})t' - (k_y b_{22} - \Omega b_{42})y' - (k_y b_{21} - \Omega b_{41})x' + \theta] \\ B_z' &= \gamma(B_{0z} + \frac{u\cos\beta}{c^2}E_{0x})* \\ &*\cos[(\Omega b_{44} - k_y b_{24})t' - (k_y b_{22} - \Omega b_{42})y' - (k_y b_{21} - \Omega b_{41})x' + \theta] \end{aligned} \tag{13}$$

Second one:

$$\begin{aligned} \mathbf{E}_x' &= E_{0x}'\cos(\Omega' t' - k_x' x' - k_y' y' - k_z' z' + \theta')\mathbf{e}_x \\ \mathbf{E}_y' &= E_{0y}'\cos(\Omega' t' - k_x' x' - k_y' y' - k_z' z' + \theta')\mathbf{e}_y \\ \mathbf{B}_z' &= B_{0z}'\cos(\Omega' t' - k_x' x' - k_y' y' - k_z' z' + \theta')\mathbf{e}_z \end{aligned} \tag{14}$$

Comparing (13) and (14) we obtain:

$$\begin{aligned} E_{0x}' &= (\gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta)E_{0x} + (u\gamma\cos\alpha)B_{0z} \\ \Omega' &= \Omega b_{44} - k_y b_{24} \\ k_x' &= k_y b_{21} - \Omega b_{41} \\ k_y' &= k_y b_{22} - \Omega b_{42} \\ k_z' &= 0 \\ \theta' &= \theta \end{aligned} \tag{15}$$

The phase light speed v_p' in the rotating frame S' is:

$$v_p' = \frac{\Omega'}{\sqrt{k_x'^2 + k_y'^2}} = \frac{\Omega b_{44} - k_y b_{24}}{\sqrt{(k_y b_{21} - \Omega b_{41})^2 + (k_y b_{22} - \Omega b_{42})^2}} = c \tag{16}$$

So, the light speed in the rotating frame equals the light speed in the inertial frame, c.

We can now proceed to calculating the amplitude and the phase transformation between the inertial and the rotating frame:

$$\theta' = \theta \tag{17}$$

The general equation of the Doppler effect is:

$$\begin{aligned} \Omega' &= \Omega b_{44} - k_y b_{24} = \Omega(b_{44} - \frac{b_{24}}{c}) = \Omega(\gamma - \frac{u\gamma\cos\alpha}{c}) = \gamma(1 - \frac{u\cos\alpha}{c})\Omega \\ \gamma &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} \\ u &= r\omega \\ \alpha &= \omega\gamma\tau \\ \beta &= \omega\gamma^2\tau \end{aligned} \tag{18}$$

The general equations for aberration are:

$$\begin{aligned} k_x' &= k(\sin\alpha\cos\beta - \gamma\cos\alpha\sin\beta - \frac{u\gamma\sin\beta}{c}) \\ k_y' &= k(\sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta + \frac{u\gamma\cos\beta}{c}) \\ k_z' &= 0 \end{aligned} \tag{19}$$

In the rotating frame, the wave follows a curved trajectory. It is interesting to observe that for $\tau=0$ we have so $\alpha=\beta=0$ and:

$$\begin{aligned} k_x' &= 0 \\ k_y' &= k\gamma(1 + \frac{u}{c}) \\ k_z' &= 0 \end{aligned} \tag{20}$$

meaning that the wave vector in frame S' starts in the same direction as the wave vector in frame S before it starts to progressively curve away as the proper time τ increases.

4. GENERAL CASE OF ROTATION ABOUT AN ARBITRARY AXIS

In a prior paper we have shown that the particular transformation (1) can be generalized for the case of arbitrary direction. The general case is treated by transforming the problem into the particular case treated in [1] through a transformation into the "canonical case", followed by an application of the transformation from the rotating frame into the inertial frame , ending with the inverse of the first transformation, as shown below:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = (Rr^{-1} * Phy\_rotation * Rr) \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{21}$$

$$Rr = Rot(\mathbf{e}_x)_{90^0} * Rot(\mathbf{e}_y)_{90^0-\varphi} * Rot_y \tag{22}$$

$Rot(\mathbf{e}_y)_{90^0-\varphi} * Rot_y$ aligns ω with e_y. The second step is comprised by another rotation around the x-axis by 90^0 that aligns ω with e_y:

$$Rot(\mathbf{e}_x)_{90^0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Expression (21) gives the solution for the general case, of arbitrary angular velocity direction. Re-writing (6) and (7) in tensor form we get:

$$\begin{bmatrix} E'_x & B'_x & * & * \\ E'_y & * & B'_y & * \\ * & E'_z & * & * \\ * & * & * & B'_z \end{bmatrix} =$$

$$\begin{bmatrix} \gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta & \gamma \mathrm{sim}\alpha\cos\beta - \cos\alpha\sin\beta & u\gamma\cos\alpha & 0 \\ \gamma\cos\alpha\sin\beta - \sin\alpha\cos\beta & \gamma\sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta & u\gamma\sin\alpha & 0 \\ u\gamma\cos\beta & u\gamma\sin\beta & 0 & \gamma \\ 0 & \dfrac{u\gamma\cos\beta}{c^2} & \dfrac{u\gamma\sin\beta}{c^2} & \gamma \end{bmatrix}$$

$$\begin{bmatrix} E_x & B_x & B_y & 0 \\ E_y & B_y & B_x & E_x \\ B_z & \dfrac{E_z}{c^2} & \dfrac{E_z}{c^2} & E_y \\ 0 & E_z & 0 & B_z \end{bmatrix} \tag{24}$$

The asterisks represent entries with no particular physical meaning, we do not care about them. Then, the general transform is

$$Rr * \begin{bmatrix} \gamma\cos\alpha\cos\beta + \sin\alpha\sin\beta & \gamma \mathrm{sim}\alpha\cos\beta - \cos\alpha\sin\beta & u\gamma\cos\alpha & 0 \\ \gamma\cos\alpha\sin\beta - \sin\alpha\cos\beta & \gamma\sin\alpha\sin\beta + \gamma\cos\alpha\cos\beta & u\gamma\sin\alpha & 0 \\ u\gamma\cos\beta & u\gamma\sin\beta & 0 & \gamma \\ 0 & \dfrac{u\gamma\cos\beta}{c^2} & \dfrac{u\gamma\sin\beta}{c^2} & \gamma \end{bmatrix}$$

$$* Rr^{-1} \tag{25}$$

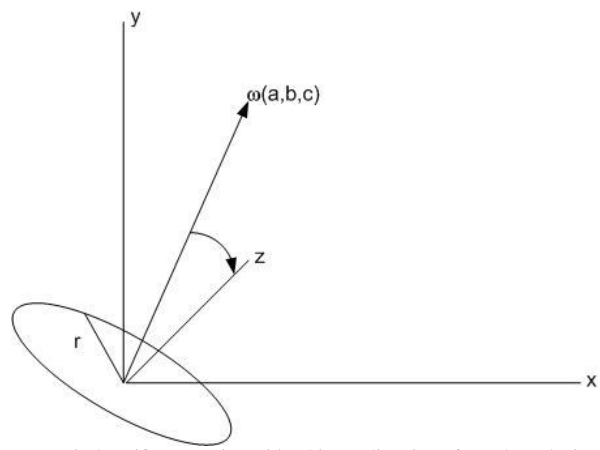

Fig.2 Uniform rotation with arbitrary direction of angular velocity

5. APPLICATION I: THE GENERAL EXPRESSIONS FOR ABERRATION AND FOR THE DOPPLER EFFECT

We have seen in section 3 that the relativistic Doppler effect can be derived from the frame invariance of the expression:

$$\Psi = \Omega t - (k_x x + k_y y + k_z z) + \varphi \tag{26}$$

We have seen in section 4 that the general coordinate transformation from an accelerated frame S' into an inertial frame S is:

$$\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}_{\omega} \begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} \tag{27}$$

The subscript ω represents the dependence of the matrix elements $a_{ij}=a_{ij}(\omega)$ of the angular velocity $\omega = (\omega_x, \omega_y, \omega_z)$ between frames S and S'. Substituting (27) into (26) we obtain:

$$\begin{aligned} \Psi &= \Omega(\mathbf{a_4}.\mathbf{x'}) - k_x(\mathbf{a}_1.\mathbf{x'}) - k_y(\mathbf{a}_2.\mathbf{x'}) - k_z(\mathbf{a}_3.\mathbf{x'}) + \varphi = \\ &= (\omega a_{44} - k_x a_{14} - k_y a_{24} - k_z a_{34})t' - (k_x a_{11} + k_y a_{21} + k_z a_{31} - \omega a_{41})x' - \\ &- (k_x a_{12} + k_y a_{22} + k_z a_{32} - \omega a_{42})y' - (k_x a_{13} + k_y a_{23} + k_z a_{33} - \omega a_{43})z' + \varphi \end{aligned} \tag{28}$$

On the other hand, in the rotating frame S':

$$\Psi' = \Omega' t' - (k_x^{'} x' + k_y^{'} y' + k_z^{'} z') + \varphi' = \Psi \tag{29}$$

Comparing (28) and (29) we obtain the general expressions of the relativistic Doppler effect between the inertial frame S and the accelerated frame S':

$$\begin{aligned} \Omega' &= \Omega a_{44} - k_x a_{14} - k_y a_{24} - k_z a_{34} \\ k_x^{'} &= k_x a_{11} + k_y a_{21} + k_z a_{31} - \Omega a_{41} \\ k_y^{'} &= k_x a_{12} + k_y a_{22} + k_z a_{32} - \Omega a_{42} \\ k_z^{'} &= k_x a_{13} + k_y a_{23} + k_z a_{33} - \Omega a_{43} \\ \varphi' &= \varphi \end{aligned} \tag{30}$$

In matrix form:

$$\begin{bmatrix} \Omega' \\ k_x^{'} \\ k_y^{'} \\ k_z^{'} \end{bmatrix} = \begin{bmatrix} a_{44} & -a_{14} & -a_{24} & -a_{34} \\ -a_{41} & a_{11} & a_{21} & a_{31} \\ -a_{42} & a_{12} & a_{22} & a_{32} \\ -a_{43} & a_{13} & a_{23} & a_{33} \end{bmatrix}_{\omega} \begin{bmatrix} \Omega \\ k_x \\ k_y \\ k_z \end{bmatrix} \tag{31}$$

For the reverse transformations we start with:

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}^{-1} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{32}$$

For simplicity, we re-write (32) as:

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}_{\omega} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \tag{33}$$

In the rotating frame S':

$$\Psi' = \Omega' t' - (k_x' x' + k_y' y' + k_z' z') + \varphi' \tag{34}$$

Substituting (33) into (34):

$$\begin{aligned} \Psi' &= \Omega'(\mathbf{b_4}.\mathbf{x}) - k_x'(\mathbf{b_1}.\mathbf{x}) - k_y'(\mathbf{b_2}.\mathbf{x}) - k_z'(\mathbf{b_3}.\mathbf{x}) + \varphi' = \\ &= (\Omega' b_{44} - k_x' b_{14} - k_y' b_{24} - k_z' b_{34})t - (k_x' b_{11} + k_y' b_{21} + k_z' b_{31} - \Omega' b_{41})x - \\ &- (k_x' b_{12} + k_y' b_{22} + k_z' b_{32} - \Omega' b_{42})y - (k_x' b_{13} + k_y' b_{23} + k_z' b_{33} - \Omega' b_{43})z + \varphi' \end{aligned} \tag{35}$$

On the other hand, in frame S:

$$\Psi = \Omega t - (k_x x + k_y y + k_z z) + \varphi = \Psi' \tag{36}$$

Comparing (35) and (36) we obtain the general expressions of the relativistic Doppler effect between the inertial frame S' and the accelerated frame S:

$$\begin{aligned} \Omega &= \Omega' b_{44} - k_x' b_{14} - k_y' b_{24} - k_z' b_{34} \\ k_x &= k_x' b_{11} + k_y' b_{21} + k_z' b_{31} - \Omega' b_{41} \\ k_y &= k_x' b_{12} + k_y' b_{22} + k_z' b_{32} - \Omega' b_{42} \\ k_z &= k_x' b_{13} + k_y' b_{23} + k_z' b_{33} - \Omega' b_{43} \\ \varphi &= \varphi' \end{aligned} \tag{37}$$

In matrix form:

$$\begin{bmatrix} \Omega \\ k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} b_{44} & -b_{14} & -b_{24} & -b_{34} \\ -b_{41} & b_{11} & b_{21} & b_{31} \\ -b_{42} & b_{12} & b_{22} & b_{32} \\ -b_{43} & b_{13} & b_{23} & b_{33} \end{bmatrix}_{\omega} \begin{bmatrix} \Omega' \\ k_x' \\ k_y' \\ k_z' \end{bmatrix} \tag{38}$$

6. APPLICATION II: THE DOPPLER EFFECT FOR ROTATING EMITTER AND RECEIVER

In this section we treat the general case, the emitter moves with angular velocity ω1 and the receiver moves with arbitrary angular velocity ω2 , both with respect to the inertial frame S. According to (38):

$$\begin{bmatrix} \Omega \\ k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} b_{44} & -b_{14} & -b_{24} & -b_{34} \\ -b_{41} & b_{11} & b_{21} & b_{31} \\ -b_{42} & b_{12} & b_{22} & b_{32} \\ -b_{43} & b_{13} & b_{23} & b_{33} \end{bmatrix}_{\omega 1} \begin{bmatrix} \Omega' \\ k_x' \\ k_y' \\ k_z' \end{bmatrix}_{emitter} \tag{39}$$

According to (31):

$$\begin{bmatrix} \Omega' \\ k_x' \\ k_y' \\ k_z' \end{bmatrix}_{receiver} = \begin{bmatrix} a_{44} & -a_{14} & -a_{24} & -a_{34} \\ -a_{41} & a_{11} & a_{21} & a_{31} \\ -a_{42} & a_{12} & a_{22} & a_{32} \\ -a_{43} & a_{13} & a_{23} & a_{33} \end{bmatrix}_{\omega 2} \begin{bmatrix} \Omega \\ k_x \\ k_y \\ k_z \end{bmatrix} \tag{40}$$

From (39) and (40) we obtain the general form of Doppler effect and aberration for the case of the emitter and the receiver moving with arbitrary angular velocities ω1 and ω2 with respect to the same inertial reference frame:

$$\begin{aligned} \begin{bmatrix} \Omega' \\ k'_x \\ k'_y \\ k'_z \end{bmatrix}_{receiver} &= \begin{bmatrix} a_{44} & -a_{14} & -a_{24} & -a_{34} \\ -a_{41} & a_{11} & a_{21} & a_{31} \\ -a_{42} & a_{12} & a_{22} & a_{32} \\ -a_{43} & a_{13} & a_{23} & a_{33} \end{bmatrix}_{\omega 2} \\ &\begin{bmatrix} b_{44} & -b_{14} & -b_{24} & -b_{34} \\ -b_{41} & b_{11} & b_{21} & b_{31} \\ -b_{42} & b_{12} & b_{22} & b_{32} \\ -b_{43} & b_{13} & b_{23} & b_{33} \end{bmatrix}_{\omega 1} \begin{bmatrix} \Omega' \\ k'_x \\ k'_y \\ k'_z \end{bmatrix}_{emitter} \end{aligned} \tag{41}$$

A quick sanity check shows that for ω1= ω2:

$$\begin{bmatrix} \Omega' \\ k_x' \\ k_y' \\ k_z' \end{bmatrix}_{receiver} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{bmatrix} \begin{bmatrix} \Omega' \\ k_x' \\ k_y' \\ k_z' \end{bmatrix}_{emitter} \tag{42}$$

In this case, there is no Doppler effect but there is obviously aberration since the terms denoted by asterisks are not null.

7. CONCLUSIONS

We constructed the general transforms from an inertial frame of reference into an uniformly rotating frame. We concluded by deriving the general form of the relativistic Doppler effect and of the relativistic aberration formulas for the case of uniformly rotating frames. The solution is of great interest for real life applications, because our earth-bound laboratories are inertial only in approximation; in real life, the laboratories are accelerated and rotate. We produced a blueprint for generalizing the solutions for the arbitrary cases and we concluded with an application that explains the general case of planar electromagnetic waves. A very interesting consequence is the fact that light speed in vacuum in the rotating frames is "c". A second interesting consequence is that rotation induces aberration.

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