A General Study of Internal Conical Refraction Phenomenon of Some Hollow Beams in Biaxial Gyrotropic and Non-Gyrotropic Crystals

Faroq Saad, Abdelmajid Belafhal
Laboratory of Nuclear, Atomic and Molecular Physics
Department of Physics, Faculty of Sciences, Chouaïb Doukkali University,
P. B 20, 24000 El Jadida, Morocco
belafhal@gmail.com

(Received 15th May, 2016; Accepted 01st June, 2016; Published: 20th June, 2016)

Abstract- In this paper, we study the effect of internal conical refraction using some hollow beams as hollow Gaussian beam (HGB) and controllable dark-hollow beam (CDHB), through biaxial gyrotropic and non-gyrotropic crystals. The analytical expressions of the amplitude and the intensity distribution of these beams in the considered crystals are derived. Some numerical simulations are also given to illustrate the variation of the intensity in radial direction versus the parameters of the incident beams as the beam order, the propagation distance Z, the gyrotropic parameter $\varepsilon_{33}$ and the ratio $r_0/\omega$ between the radius cylindrical of conical refraction $r_0$ and radius waist beams $\omega$. The results provide more general characteristics and propagation by a conical refraction of HGB and CDHB in biaxial non-gyrotropic crystals. Additionally, the propagation of the flat-topped and the Gaussian beams both in biaxial gyrotropic and non-gyrotropic crystals are deduced as particular cases of the present investigation.

Key Words: Internal conical refraction; Biaxial gyrotropic crystals; Biaxial non-gyrotropic crystals; Hollow Gaussian beam; Controllable dark-hollow beam; Flat-topped beam.

1. INTRODUCTION

The phenomenon of conical refraction was discovered theoretically by W. Hamilton, in 1832 [1], and confirmed experimentally by H. Lloyd [2] in the same year. The transformation of internal conical refraction has been described in the geometrical optics in Ref. [3]. This phenomenon results from the propagating of a monochromatic light beams along an optic axis of a biaxial crystal as a cylinder. It represents the main subject that has been studied in several works [4-19]. The effect of internal conical refraction in the crystal was introduced and studied in theory by Belsky and Khapalyuk [6], and Berry [7], and is based on two main equations. This theory was explored in detail by Belsky and Stepanov [8] and Belafhal [9], for the case of non-gyrotropic biaxial crystals. They have shown that the evolution of conical refraction depends on the ratio $r_0/\omega$ between the ring radius of cylinder and the waist radius of incident beam. These results were developed and generalized to Chiral biaxial crystals by Belsky and Stepanov [10]. They have demonstrated and compared the intensity distribution results by a numerical simulation and experimental study concerning the internal conical refraction of fundamental Gaussian beam in biaxial gyrotropic of $\alpha$-iodic acid crystals. On the other hand, Berry et al. [11, 12] have also studied the effect of chirality for two dimensionless parameters. Recently, there have appeared many studies devoted to understand the effect of conical refraction both in theory and experiment [13-19]. Among these, Peet and Zolotukhin [13] have demonstrated the transformation of circularly and linearly polarized Gaussian input beam in biaxial crystal, and circularly polarized Laguerre-Gauss input beam has also been treated [14, 15]. Additionally, the conical diffraction intensity of light beams with top-hat input beams has been studied by R. T. Darcy et al. [16]. Also, Turpin et al. [17] have also investigated polarized super-Gaussian conical refraction beam in biaxial crystals. More recently, Turpin et al. [18] have studied the transformation of light beams through a biaxial crystal by using a new formalism able to show the transition of double refraction, and also used this formalism for the propagation of non-homogeneously polarized beams through a biaxial crystal. The transformation of radialy and azimuthally polarized input light beam has also been demonstrated in Ref. [19].

The theory of the intensity distribution of internal conical refraction for a light beam passing through a biaxial gyrotropic crystal is more complicated than it is in the case of a non-gyrotropic crystal. Our aim here is to investigate a theoretical study of conical refraction phenomenon for a new kind of hollow beams as: HGB and CDHB, respectively. The present paper is organized as follows: In section 2, we present a theoretical general description of the transformation of light beams in gyrotropic biaxial crystals by internal conical refraction. In section 3, an analytical expression, to describe the transformation of HGB in a biaxial of gyrotropic crystal, is studied, and some particular cases will be treated and discussed from the presented theory. In section 4, the transformation of CDHB and flat-topped beam, propagating in a biaxial gyrotropic and non-gyrotropic crystal, are also established. In section 5, several numerical examples are also given to illustrate our analytical results. Finally, we conclude our work in last section.

2. INTERNAL CONICAL REFRACTION FOR A LIGHT BEAM IN BIAXIAL GYROTROPIC CRYSTALS

In gyrotropic of biaxial crystals, the intensity distribution of internal conical refraction is investigated. The incident
beam is propagated perpendicularly along to one of the optic axes as a cylinder. Fig. 1 shows the schematic view of the experimental setup to study the conical diffraction phenomenon by using a biaxial crystal.

Now, we consider the plane of a conical refraction ring in the polar coordinates system as: \( r = r_0 \) and \( \phi = \phi_0 \), where \( r_0 \) is the radial component in cylindrical coordinate, \( \phi_0 \) is the azimuth angle and \( r_0 \) is the radius of the cylinder of light emerging on a biaxial crystal. The emerging light of an electric field of beam by internal conical refraction in biaxial gyrotropic crystals has two components given by [10]

\[
E_i^r = D[k_i B_i^r (r) + (k_i \cos \phi + k_s \sin \phi)B_i(r)] \\
\quad - k_i B_i^s (r)]\exp(ikn_0 z),
\]

\[
E_i^s = D[k_i B_i^s (r)]\exp(ikn_0 z),
\]

and

\[
E_i^r = D[k_i B_i^r (r) - (k_i \cos \phi - k_i \sin \phi)B_i(r)] \\
\quad + k_i B_i^s (r)]\exp(ikn_0 z),
\]

where \( k_i = \cos \gamma, \), \( k_s = \sin \gamma \) with \( \gamma \) is the angle of polarized incident beam, and \( B_i, B_s \) and \( B_s \) are the field radial function dependence in the conical refraction ring and defining by integrated expressions as follow

\[
B_i(r) = \frac{1}{2\pi} \int A(\rho) \exp \left( -\frac{ikZ\rho^2}{2} \right) \sin(k_0 \sqrt{\rho^2 + g_{33}^2})d\rho,
\]

\[
B_s(r) = \frac{1}{2\pi} \int A(\rho) \exp \left( -\frac{ikZ\rho^2}{2} \right) \cos(k_0 \sqrt{\rho^2 + g_{33}^2})d\rho,
\]

and

\[
B_i(r) = \frac{g_{33}}{2\pi} \int A(\rho) \exp \left( -\frac{ikZ\rho^2}{2} \right) \cos(k_0 \sqrt{\rho^2 + g_{33}^2})d\rho,
\]

\[
B_s(r) = \frac{g_{33}}{2\pi} \int A(\rho) \exp \left( -\frac{ikZ\rho^2}{2} \right) \sin(k_0 \sqrt{\rho^2 + g_{33}^2})d\rho,
\]

where

\[
Z = \frac{\pi - \Delta}{n_0},
\]

\[
r_0 = \frac{\alpha}{2\epsilon_2},
\]

\[
g_{33} = \frac{\pi}{\alpha} g_{33},
\]

\[
D = \frac{4n_0 \pi}{(n_0 + \pi)\eta} \exp \left( ik(\pi - n_0 - \frac{1}{8\pi} g_{33}^2) \right),
\]

\[
\pi = \sqrt{\epsilon_1},
\]

\[
\Delta = \frac{n_0}{\pi} \left( \frac{\pi - n_0}{2\epsilon_2} + \frac{\pi - n_0}{n_0} \right),
\]

and

\[
\alpha = \frac{\pi}{\epsilon_1} \left( \frac{\pi - n_0}{\epsilon_2} \right),
\]

In Eqs. 2, \( J_{\frac{3}{2}} \) are the zeroth and the first order Bessel functions, \( Z \) is the propagation distance which is measured from the focal plane, \( r_0 \) is the radius of cylinder of light emerging from the crystal, \( g_{33} \) is the gyrotropic parameter, \( n_0 \) is the index of refraction and \( \epsilon_1 > \epsilon_2 > \epsilon_3 \) represent the principal values of the dielectric tensor, and

\[
A(\rho) = 2\pi k \int f(\rho) J_0(\rho k) r dr,
\]

where \( f(\rho) \) is the electric field distribution of the considered laser beam and \( k \) is the wave number. This last equation gives the Fourier image of the field distribution in the incident beam cross-section.

From Eqs. (1-a), (1-b), (2-a), (2-b) and (2-c) which describe the outgoing beam dependence of the azimuth angle \( \phi \), the intensity distribution of the outgoing field is given by

\[
I_r = |D\left[ |B_i|^2 - |B_s|^2 \right] + 2\text{Re}(B_i B_s^*) \cos(\phi - 2\gamma) + 2\text{Re}(B_i B_s^*) \sin(\phi - 2\gamma)|.
\]

This last equation represents the intensity distribution in biaxial gyrotropic crystals by conical refraction when it turns through the angle \( 2\gamma \) around the polar coordinate system.

Now, we will discuss the intensity distribution of the internal conical refraction for some hollow beams propagating in biaxial gyrotropic crystals by considering a hollow Gaussian and Controllable dark-hollow distributions in section 3 and section 4, respectively.

3. HOLLOW GAUSSIAN FIELD IN BIAXIAL GYROTROPIC CRYSTALS

In this section, we give the intensity distribution of the internal conical refraction for the case of HGB in biaxial gyrotropic crystals. Assuming that, the electric field distribution of HGB in the cylindrical coordinate system at the source plane \( z=0 \), expressed by [20]

\[
f(\rho) = G_0 (\rho r)^{\beta-1} \exp(-\beta r^2), \quad N = 0,1,2,\ldots,
\]

where \( G_0 \) is a constant representing the amplitude coefficient, \( N \) is the order of the HGB, and \( \beta=1/\omega^2 \) with \( \omega \) is the Gaussian waist at the plane \( z=0 \). Also, HGB can be expressed as a superposition of a series of Laguerre-Gaussian modes as
\[ f(r) = \frac{G_s}{2} \exp(-\beta r^2) \sum_{n} (-1)^n \left[ \frac{1}{2} \right]_{m} L_{m}(\alpha r^2) \]  

where \( L_m(\cdot) \) denotes Laguerre polynomial with mode order \( m \), \( \left[ \cdot \right]_{m} \) is a binomial coefficient, and \( \alpha = 2\beta \).

By substituting Eq. (7) into Eq. (4) and using the following integral formula [21]

\[ \int_{0}^{\infty} x^{n-1} e^{-x^2} L_{m}(\alpha' x^2) \, dx = 2^{-\frac{n}{2}} \beta^{-\frac{n+1}{2}} (\beta - \alpha')^{-\frac{1}{2}} \left( \frac{\alpha}{\alpha' - \beta} \right) \alpha^\frac{n}{2} L_{m-\frac{1}{2}} \left( \frac{\alpha}{\alpha' - \beta} \right) \]

and after tedious integral calculation, Eq. (4) becomes

\[ A(\rho) = \frac{G_s}{2} \sum_{n} (-1)^n \left[ \frac{1}{2} \right]_{m} L_{m}(\alpha' \rho^2) \]  

By introducing Eq. (9) into Eqs. (2-a), (2-b), and (2-c), the outgoing field of HGB after passing through a biaxial gyrotropic crystal in conical refraction can be written as

\[ B_0(r) = \frac{k' \alpha' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) L_{m}(\frac{k' \omega' \rho^2}{2}) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

\[ J_0(k r \rho) \rho^2 d\rho, \]

and

\[ B_0(r) = \frac{k' \alpha' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) L_{m}(\frac{k' \omega' \rho^2}{2}) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

\[ J_0(k r \rho) \rho^2 d\rho, \]

These last integrals converge and can be evaluated numerically. They represent the main first results of this study which describe the transformation of HGB in biaxial gyrotropic crystals by internal conical refraction.

3.1. Particular Cases

In the previous section, we have demonstrated that the intensity distribution of the internal conical refraction for a HGB can be propagated in biaxial gyrotropic crystals. From our finding established in Eqs. (10-a), (10-b) and (10-c), one can deduce particular cases by choosing some values of \( g_3 \) and \( N \).

3.1.1. Case of hollow Gaussian field through a biaxial non-gyrotropic crystal: This case is obtained when \( g_3 = 0 \).

Consequently, Eqs. (10-a), (10-b) and (10-c) reduce to

\[ B_0(r) = \frac{k' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) L_{m}(\frac{k' \omega' \rho^2}{2}) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

\[ J_0(k r \rho) \rho^2 d\rho, \]

and

\[ B_0(r) = \frac{k' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) L_{m}(\frac{k' \omega' \rho^2}{2}) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

These equations describe the field distribution of the internal conical refraction of HGB in biaxial non-gyrotropic crystals. Note that for the case of non-gyrotropic crystal \( B_0 \) vanishes and the intensity distribution does not depend on the azimuth angle \( \varphi \). So, Eq. (5) can be rewritten as:

\[ I(r) = |D|^2 \left| B_0 \right|^2 \]  

This equation represents the intensity distribution of the internal conical refraction for an incident hollow Gaussian beam in biaxial non-gyrotropic crystals.

3.1.2. Case of Gaussian distribution through a biaxial gyrotropic crystal: In this particular case, when \( N=0 \), Eqs. (10-a), (10-b) and (10-c) get the form

\[ B_0(r) = \frac{k' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) L_{m}(\frac{k' \omega' \rho^2}{2}) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

\[ J_0(k r \rho) \rho^2 d\rho, \]

and

\[ B_0(r) = \frac{k' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) L_{m}(\frac{k' \omega' \rho^2}{2}) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

which represents the electrical field of the internal conical refraction for a Gaussian beam through a biaxial gyroamorphic crystal. Note that these equations are the same as Eqs. (15) of Ref. [10].

3.1.3. Case of Gaussian field through a biaxial non-gyrotropic crystal: For this special case when \( g_3 = 0 \), and \( N=0 \), Eqs. (10-a), (10-b) and (10-c) simplify to

\[ B_0(r) = \frac{k' \omega N!}{2} \sum_{n} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \left[ \frac{1}{2} \right]_{m} \int \exp \left( -\frac{k' \omega' \rho^2}{4} \right) \times \exp \left( -\frac{i k Z \rho^2}{2} \right) \frac{\sin(k r \sqrt{\rho^2 + \frac{1}{2} + B_0})}{\sqrt{\rho^2 + \frac{1}{2} + B_0}} \right) \]  

\[ J_0(k r \rho) \rho^2 d\rho, \]
$B_r (r) = \frac{k^2 \omega^2}{2} \int_0^\infty \exp \left( -\frac{k^2 \omega^2 \rho^2}{4} - \frac{ikZ\rho^2}{2} \right) \times \sin(kr,\rho) J_0(kr) d\rho$. 

These last equations give the analytical formula of the electric field of a Gaussian beam propagating through a biaxial non-gyrotropic crystal. These relations are in accordance with results of Refs. [8,9].

4. CONTROLLABLE DARK-HOLLOW DISTRIBUTION OF CIRCULAR SYMMETRY IN BIAXIAL GYROTROPIC CRYSTALS

In this section, we study the intensity distribution of the internal conical refraction for an interesting kind of hollow beam called Controllable dark-hollow beam (CDHB), passing through a biaxial gyrotropic crystal. The field distribution of this beam in the cylindrical coordinates system at input plane is defined as follows [22]

$$f(r) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{N} \left[ \exp \left( -\frac{nr^2}{\rho^2} \right) - \exp \left( -\frac{nr^2}{p\rho^2} \right) \right],$$

where $\binom{N}{n}$ is a binomial coefficient, $N$ is the order of the CDHB, $\omega$ is the width beam and $p$ is a real positive parameter ($p<1$). Also, the area of the dark region at the centre can be controlled by the parameter $p$.

By inserting Eq. (15) into Eq. (4) and recalling the integral formulae [21],

$$\int e^{z_1+2 \cdot e^{-z_2}} J_n(z_1) dt = \frac{e^{z_1+2 \cdot e^{-z_2}}}{(2a)^{1/2}} e^{z_1+2 \cdot e^{-z_2}},$$

with $Re \omega>0$, $Re \omega>1$, and after tedious integral calculation, Eq. (4) can be written as

$$A(\rho) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{N} \frac{n!}{n!(N-n)!} \frac{p^k \omega^2}{n!} \left[ \exp \left( -\frac{k^2 \omega^2 \rho^2}{4n} \right) - \exp \left( -\frac{p^k \omega^2 \rho^2}{4n} \right) \right].$$

By introducing Eq. (17) into Eqs. (2-a), (2-b) and (2-c), we can express the outgoing field of a CDHB after passing a biaxial gyrotropic crystal as

$$B_r (r) = \frac{k^2 \omega^2}{2N} \sum_{n=1}^\infty \frac{(-1)^{n+1} N!}{n!(N-n)!} \left[ \exp \left( -\frac{k^2 \omega^2 \rho^2}{4n} \right) - \exp \left( -\frac{p^k \omega^2 \rho^2}{4n} \right) \right].$$

These equations describe the field distribution of the conical refraction for a CDHB after passing through a biaxial non-gyrotropic crystal.

4.1. Particular Cases

In the above section, we have investigated the intensity distribution of the internal conical refraction for a CDHB passing through biaxial gyrotropic crystals. From our results established in Eqs. (18-a), (18-b) and (18-c), some particular cases can be obtained by choosing some values of $\overline{g}_{33}$, N and p.

4.1.1. Case of Controllable dark hollow distribution through a biaxial non-gyrotropic crystal: From Eqs. (18-a), (18-b) and (18-c), by putting $\overline{g}_{33} = 0$, one obtains

$$B_r (r) = \frac{k^2 \omega^2}{2N} \sum_{n=1}^\infty \frac{(-1)^{n+1} N!}{n!(N-n)!} \times \left[ \exp \left( -\frac{k^2 \omega^2 \rho^2}{4n} \right) - \exp \left( -\frac{p^k \omega^2 \rho^2}{4n} \right) \right].$$

4.1.2. Case of flat-topped distribution through a biaxial gyrotropic crystal: For this case, by taking $N>1$ and $p=0$ in Eq. (15), the incident beam obtaining corresponds to the expression of the electric field with circular flat-topped beam which is given by [23]

$$f(r) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{N} \left[ \exp \left( -\frac{nr^2}{\omega^2} \right) \right],$$

and Eqs. (18-a), (18-b) and (18-c) reduce to
The integration limit is 4.1.3. Case of flat-topped distribution through a biaxial non-gyrotropic crystal in conical refraction for HGB and CDHB through biaxial non-gyrotropic crystals which are calculated and expressed by analytical expressions in section 3, we will give some numerical simulations for different beam orders \( N = 0 \) and 1 at \( Z = 0, 10, 30 \) mm, which are presented in Figs. (2-5), obtained by using the parameters previously mentioned.

In Figs. (2-5), we display the intensity profiles of the internal conical refraction of HGB, in biaxial gyrotropic crystals which are calculated by the above analytical results elaborated in Eqs. (5), (10-a), (10-b) and (10-c), for two values of the beam order \( N \) at different propagation distances \( Z \) with the polarized incident beam \( (\gamma=0) \) and \( \widetilde{g}_{33} = 0.0045. \) From Fig. 2, the intensity distribution of HGB for beam order \( N \) equal to zero varies with different propagation distances \( Z \) (see Fig. 2 (a-c)). For \( Z=0 \), it can be shown that the spiral phase shape of the rings is observed and takes on a clock-wise spiral distribution (see Fig. 2 (a)). On increasing the propagation distance \( Z \), the bright spot becomes more observable and the spiral rings also increases. Furthermore, the axial spike appears as \( Z \) increases (see Fig. 2 (c)). For this case, the obtained results correspond to the fundamental Gaussian beam in biaxial gyrotropic crystals. Fig. 3 presents the intensity profile of HGB as Fig. 2, but here for the beam order \( N \) is equal to one. We can show from these figures of different values of \( Z \), that the spiral of the rings is observed and increases. Additionally, the bright spot of the intensity distribution becomes larger and the axial spike is shown (see Fig. 3 (a-c)). Generally, one can see that the bright spot and the spiral rings can be changed when \( N \) and \( Z \) are increasing (see Figs. 2-3).

Figs. 4 and 5 give the results of numerical simulations established in the above case (3.1.1). These results correspond to intensities calculated from Eqs. (11-a), (11-b) and (12), which describe the transformation of hollow Gaussian beam in a biaxial non-gyrotropic crystal for different values of beam order \( N \) and choose the parameters which are used in the previous figures.

Fig. 4 illustrates the intensity profile of HGB with order \( N=0 \) at different propagation distances \( Z \). When \( Z=0 \), the intensity of this beam presents a central dark spot and double bright rings separated by dark ring (see Fig. 4 (a)). As \( Z \) increases the separation of rings is shown, and the central dark spot decreases gradually. Then, the bright spot appears at the centre with further increases of propagation distances \( Z \) (see Fig. 4 (c)). In this case, the results, obtained are identical to the results for the case of the fundamental Gaussian beam in Ref. [13].

Fig. 5 gives the intensity profile of HGB as Fig. 4, but in this time for the beam order \( N \) is equal to one. From Fig. 5 (a), we can see that at \( Z=0 \), the intensity gets several bright rings with different intensities surrounding the central dark spot. On increasing the propagation distance \( Z \), the dark spot reduces
Fig. 2: The intensity distribution of the internal conical refraction for a HGB in gyrotropic crystals for beam order $N=0$ at (a) $Z=0$, (b) $Z=10\text{mm}$ and (c) $Z=30\text{mm}$.

Fig. 3: The intensity distribution of the internal conical refraction for a HGB in gyrotropic crystals for beam order $N=1$ at (a) $Z=0$, and (b) $Z=10\text{mm}$.
Fig. 3c: The intensity distribution of the internal conical refraction for a HGB in gyrotropic crystals for beam order $N=1$ at $Z=30\text{mm}$.

Fig. 4: Intensity distribution of HGB in biaxial non-gyrotropic crystals computed for beam order $N=0$ at (a) $Z=0$, (b) $Z=10\text{mm}$ and (c) $Z=30\text{mm}$.

Fig. 5a: Intensity distribution of HGB in biaxial non-gyrotropic crystals computed for beam order $N=1$ with $r_0=240\mu\text{m}$, $\omega=34\mu\text{m}$ and $Z=0$. 
Fig. 5: Intensity distribution of HGB in biaxial non-gyrotropic crystals computed for beam order $N=1$ with $r_0=240\,\mu$m, $\omega_0=34\,\mu$m and (b) $Z=10\,\text{mm}$ and (c) $Z=30\,\text{mm}$.

Gradually and the central intensity with bright spot becomes more observable (see Fig. 5 (b) and (c)). Generally, when the beam order $N$ is increasing at some propagation distances $Z$, the dark spot reduces gradually, and the intensity of the bright rings is formed and also changes.

5.2. Case of the Controllable dark-hollow beam

The intensity profiles of CDHB in biaxial gyrotropic and non-gyrotropic crystals are established and given by analytical expressions in the above section 4. So, we will present some numerical simulations examples to illustrate our analytical results which are presented in Figs. (6-11).

Figs. 6 and 7 give the intensity distribution of the internal conical refraction for a CDHB in biaxial gyrotropic crystals calculated from Eqs. (5), (18-a), (18-b) and (18-c), with the same parameters used in Fig.2 and $p=0.9$, for two values of beam order $N$ (=1 and 2) and propagation distances $Z$= (0, 10, 30 mm).

In Fig. 6, we illustrate the intensity profile of conical refraction ring for a CDHB with the order $N$ is equal to one at different propagation distance $Z$. As $Z=0$, the bright spot of the intensity distributions is observed and the phase shape takes on a clock-wise spiral distribution (see Fig. 6 (a)). As $Z$ increases, the spiral of the rings becomes fainter (see Fig. 6 (b)). Additionally, one can show that this spiral disappears and the axial intensity spike is formed as $Z$ increases (see Fig. 6 (c)).

When the beam order $N$ is increasing at some propagation distances $Z$, the variations of intensity of conical refraction ring are shown (see Fig. 7). The simulations results of these figures are similar to those of Fig. 6. One can see from that the bright spot and the spiral of the ring can be adjusted by changing $N$ and $Z$.

Fig. 6a: Intensity profile of conical refraction ring of CDHB in biaxial gyrotropic crystals for beam order $N=1$ at $Z=0$. 
Fig. 6: Intensity profile of conical refraction ring of CDHB in biaxial gyrotropic crystals for beam order $N=1$ at (b) $Z=10\text{mm}$ and (c) $Z=30\text{mm}$.

Fig. 7: Intensity profile of conical refraction ring of CDHB in biaxial gyrotropic crystals for beam order $N=2$ at (a) $Z=0$, (b) $Z=10\text{mm}$ and (c) $Z=30\text{mm}$.
Fig. 8: Intensity distribution of internal conical refraction for a CDHB in biaxial non-gyrotropic crystals for beam order $N=1$, $p=0.9$ and (a) $Z=0$, (b) $Z=10\text{mm}$ and (c) $Z=30\text{mm}$.

Fig. 9: Intensity distribution of CDHB in biaxial non-gyrotropic crystals computed for beam order $N=2$ at (a) $Z=0$ and (b) $Z=10\text{mm}$.
**Fig. 9c:** Intensity distribution of CDHB in biaxial non-gyrotropic crystals computed for beam order $N=2$ at $Z=30\text{mm}$.

**Fig. 10:** Intensity profile of flat-topped beam through biaxial gyrotropic crystals in conical refraction for beam order $N=2$ at (a) $Z=0$, (b) $Z=10\text{mm}$ and (c) $Z=30\text{mm}$.

**Fig. 11a:** Intensity profile of flat-topped beam through biaxial non-gyrotropic crystals in conical refraction for beam order $N=2$ and $Z=0$. 
Figs. 8 and 9 present the numerical results of the intensity distribution of the conical refraction for a CDHB passing through biaxial non-gyrotropic crystals which computed in the above case (4.1.1) by the use of Eqs. (12), (19-a) and (19-b), respectively, for two beam orders \( N = 1 \) and \( 2 \) at propagation distances \( Z = 0, 10, 30 \text{ mm} \) and used the same parameters as those for the previous figures. Fig. 8 gives the intensity profile for beam order \( N=1 \) at some propagation distances \( Z \). One can see that, at \( Z=0 \), the central intensity of this beam keeps the dark spot surrounded by several bright rings which are formed with different intensities. As \( Z \) increases, the dark spot decreases gradually. Then, the bright spot also appears at the centre. With further increases of the beam order \( N \) at propagation distances \( Z \), the variation of intensity profile is shown (see Fig. 9). The numerical results of these last figures are similar to those of Fig. 8, and we can see from that the dark spot and the intensity of the bright rings can be changed by increasing \( N \) and \( Z \).

Fig. 10 gives the intensity distribution of the internal conical refraction for a flat-topped beam propagating in a biaxial gyrotropic crystal which is elaborated in the above case (4.1.2) by Eqs. (5), (21-a), (21-b) and (21-c) for a beam order \( N = 2 \) and for three values of propagation distances \( Z = 0, 10, 30 \text{ mm} \).

One can see that, Fig. 10 is similar to Figs. 6 and 7, but in this case for \( p=0 \) and \( N=1 \). The intensity profile of this beam gets also its spiral shape of the rings. Then, this spiral disappears and the bright spot appears at the centre as \( Z \) increases (see Fig. 10).

Fig. 11 gives the radial distributions of the internal conical refraction for a flat-topped beam passing through a biaxial non-gyrotropic crystal calculated in the particular case (4.1.3) by the use of Eqs. (12), (22-a) and (22-b).

One can see that the results obtained in this figure are similar to those of Figs. 8 and 9, but this time for \( N>1 \) and the parameter \( p \) is equal to zero. It’s clear from that, when \( Z=0 \), the central dark spot with two bright rings are observed. Additionally, the dark spot and the intensity of double bright rings are formed and they can be changed by increasing \( Z \) (see Fig. 11).

### 6. CONCLUSION

In summary, we have investigated the propagation of some hollow beams like: HGB and CDHB in a biaxial gyrotropic and non-gyrotropic crystal by internal conical refraction. Analytical expressions of the intensity distribution of these beams passing through a transparent slab of a biaxial gyrotropic crystal cut perpendicularly to one of its wave axes are derived. The simulations results of the intensity distribution of these beams are drawn. The influence of some parameters of these incident beams, the propagation distance \( Z \), the gyrotropic parameter \( \tilde{G}_{13} \) and the ratio between the ring radius of the conical refraction cylinder \( r_0 \) and the waist radius \( \omega \) are discussed. Our study generalizes the results previously established in the investigation on the internal conical refraction of HGB and CDHB through a biaxial non-gyrotropic crystal. Finally, the flat-topped and the fundamental Gaussian beams propagating, in biaxial crystals gyrotropic or non-gyrotropic, are also deduced as particular cases in our finding.

### ACKNOWLEDGMENT

The first author was supported by the Ministry of higher Education and Scientific Research of Yemen.

### REFERENCES


